Number Theory

This broad category is a popular source for GRE questions. At first, students often struggle with these problems since they have forgotten many of the basic properties of arithmetic. So, before we begin solving these problems, let's review some of these basic properties.

➢ ***"The remainder is r when p is divided by k"* means $p = kq + r$; the integer q is called the quotient. For instance, "*The remainder is 1 when 7 is divided by 3*" means $7 = 3 \cdot 2 + 1$. Dividing both sides of $p = kq + r$ by k gives the following alternative form $p/k = q + r/k$.**

Example 1: The remainder is 57 when a number is divided by 10,000. What is the remainder when the same number is divided by 1,000?

(A) 5
(B) 7
(C) 43
(D) 57
(E) 570

Since the remainder is 57 when the number is divided by 10,000, the number can be expressed as $10{,}000n + 57$, where n is an integer. Rewriting 10,000 as 1,000(10) yields

$$1{,}000(10)n + 57 =$$

$$1{,}000(10n) + 57 =$$

Now, since n is an integer, $10n$ is an integer. Letting $10n = q$, we get

$$1{,}000q + 57 =$$

Hence, the remainder is still 57 (by the $p = kq + r$ form) when the number is divided by 1,000. The answer is (D).

Method II (Alternative form)
Since the remainder is 57 when the number is divided by 10,000, the number can be expressed as $10{,}000n + 57$. Dividing this number by 1,000 yields

$$\frac{10{,}000n+57}{1000} =$$

$$\frac{10{,}000n}{1{,}000} + \frac{57}{1{,}000} =$$

$$10n + \frac{57}{1{,}000}$$

Hence, the remainder is 57 (by the alternative form $p/k = q + r/k$), and the answer is (D).

➢ **A number n is even if the remainder is zero when n is divided by 2: $n = 2z + 0$, or $n = 2z$.**

➢ **A number n is odd if the remainder is one when n is divided by 2: $n = 2z + 1$.**

- The following properties for odd and even numbers are very useful—you should memorize them:

even   *even* = *even*
odd   *odd* = *odd*
even   *odd* = *even*

even + *even* = *even*
odd + *odd* = *even*
even + *odd* = *odd*

Example 2: If n is a positive integer and $(n + 1)(n + 3)$ is odd, then $(n + 2)(n + 4)$ must be a multiple of which one of the following?

(A) 3 (B) 5 (C) 6 (D) 8 (E) 16

$(n + 1)(n + 3)$ is odd only when both $(n + 1)$ and $(n + 3)$ are odd. This is possible only when n is even. Hence, $n = 2m$, where m is a positive integer. Then,

$(n + 2)(n + 4) =$
$(2m + 2)(2m + 4) =$
$2(m + 1)2(m + 2) =$
$4(m + 1)(m + 2) =$
4   (product of two consecutive positive integers, one which must be even) =
4   (an even number), and this equals a number that is at least a multiple of 8

Hence, the answer is (D).

- Consecutive integers are written as x, $x + 1$, $x + 2$, . . .

- Consecutive even or odd integers are written as x, $x + 2$, $x + 4$, . . .

- The integer zero is neither positive nor negative, but it is even: 0 = 2   0.

- A ***prime number*** is an integer that is divisible only by itself and 1.

  The prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, . . .

- A number is divisible by 3 if the sum of its digits is divisible by 3.

  For example, 135 is divisible by 3 because the sum of its digits (1 + 3 + 5 = 9) is divisible by 3.

- A ***common multiple*** is a multiple of two or more integers.

  For example, some common multiples of 2 and 5 are 0, 10, 20, 40, and 50.

- The ***least common multiple*** (LCM) of two integers is the smallest positive integer that is a multiple of both.

  For example, the LCM of 4 and 10 is 20. The standard method of calculating the LCM is to prime factor the numbers and then form a product by selecting each factor the greatest number of times it occurs. For 4 and 10, we get

  $4 = 2^2$
  $10 = 2 \cdot 5$

  In this case, select 2^2 instead of 2 because it has the greater number of factors of 2, and select 5 by default since there are no other factors of 5. Hence, the LCM is $2^2 \cdot 5 = 4 \cdot 5 = 20$.

  For another example, let's find the LCM of 8, 36, and 54. Prime factoring yields

  $8 = 2^3$
  $36 = 2^2 \cdot 3^2$
  $54 = 2 \cdot 3^3$

  In this case, select 2^3 because it has more factors of 2 than 2^2 or 2 itself, and select 3^3 because is has more factors of 3 than 3^2 does. Hence, the LCM is $2^3 \cdot 3^3 = 8 \cdot 27 = 216$.

A shortcut for finding the LCM is to ust eep adding the largest number to itself until the other numbers divide into it evenly. For 4 and 10, we would add 10 to itself 10 + 10 = 20. ince 4 divides evenly in to 20, the LCM is 20. For 8, 36, and 54, we would add 54 to itself 54 + 54 + 54 + 54 = 216. ince both 8 and 36 divide evenly into 216, the LCM is 216.

- The absolute value of a number, , is always positive. n other words, the absolute value symbol eliminates negative signs.

  For example, $|-7| = 7$ and $|-\pi| = \pi$. Caution, the absolute value symbol acts only on what is inside the symbol, $|\ |$. For example, $-|-(7-\pi)| = -(7-\pi)$. Here, only the negative sign inside the absolute value symbol but outside the parentheses is eliminated.

Example 3: The number of prime numbers divisible by 2 plus the number of prime numbers divisible by 3 is

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

A prime number is divisible by no other numbers, but itself and 1. Hence, the only prime number divisible by 2 is 2 itself and the only prime number divisible by 3 is 3 itself. Hence, The number of prime numbers divisible by 2 is one, and the number of prime numbers divisible by 3 is one. The sum of the two is 1 + 1 = 2, and the answer is (C).

Example 4: If 15 + 16 = 0, then $15|\ |$ equals which one of the following?

(A) 15 (B) 16 (C) 15 (D) 16 (E) 16

olving the given equation 15 + 16 = 0 yields = 16 15.

ubstituting this into the expression $15|\ |$ yields

$$15|\ | = 15\left|\frac{-16}{15}\right| = 15\left(\frac{16}{15}\right) = 16$$

The answer is (D).

- The product (uotient) of positive numbers is positive.

- The product (uotient) of a positive number and a negative number is negative.

  For example, 5(3) = 15 and $\frac{6}{3} =\ 2$.

- The product (uotient) of an even number of negative numbers is positive.

  For example, (5)(3)(2)(1) = 30 is positive because there is an even number, 4, of positives. $\frac{9}{2} = \frac{9}{2}$ is positive because there is an even number, 2, of positives.

- The product (uotient) of an odd number of negative numbers is negative.

  For example, $(-2)(-\pi)(-\sqrt{3}) = -2\pi\sqrt{3}$ is negative because there is an odd number, 3, of negatives. $\frac{(-2)(-9)(-6)}{(-12)\ {-18}/{2}} = -1$ is negative because there is an odd number, 5, of negatives.

- The sum of negative numbers is negative.

  For example, 3 5 = 8. ome students have trouble recogni ing this structure as a sum because there is no plus symbol, +. But recall that subtraction is defined as negative addition. o 3 5 = 3 + (5).

- A number raised to an even exponent is greater than or e ual to zero.

  For example, $(-\pi)^4 = \pi^4 \geq 0$, and $^2 \geq 0$, and $0^2 = 0 \cdot 0 = 0 \geq 0$.