Functions

DEFINITION

A function is a special relationship (correspondence) between two sets such that for each element *x* in its domain there is assigned one and only one element *y* in its range.

Notice that the correspondence has two parts:

1) For each *x* there is assigned *one y*. (This is the ordinary part of the definition.)
2) For each *x* there is assigned *only one y*. (This is the special part of the definition.)

The second part of the definition of a function creates the uniqueness of the assignment: There cannot be assigned two values of *y* to one *x*. In mathematics, uniqueness is very important. We know that 2 + 2 = 4, but it would be confusing if 2 + 2 could also equal something else, say 5. In this case, we could never be sure that the answer to a question was the *right* answer.

The correspondence between *x* and *y* is usually expressed with the function notation: $y = f(x)$, where *y* is called the dependent variable and *x* is called the independent variable. In other words, the value of *y* depends on the value of *x* plugged into the function. For example, the square root function can be written as $y = f(x) = \sqrt{x}$. To calculate the correspondence for *x* = 4, we get $y = f(4) = \sqrt{4} = 2$. That is, the square root function assigns the unique *y* value of 2 to the *x* value of 4. Most expressions can be turned into functions. For example, the expression $2\ \ -\frac{1}{\ }$ becomes the function

$$f(\ ) = 2\ \ -\frac{1}{\ }$$

DOMAIN AND RANGE

We usually identify a function with its correspondence, as in the example above. However, a function consists of three parts: a domain, a range, and correspondence between them.

➢ **The *domain* of a function is the set of *x* values for which the function is defined.**

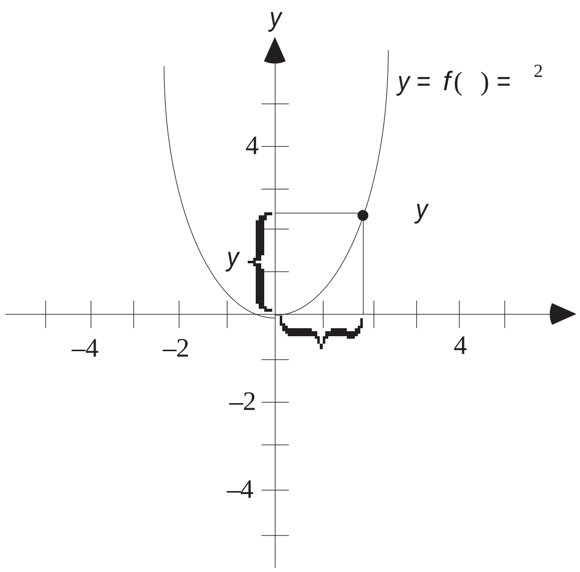
For example, the function $f(\ ) = \frac{1}{\ -1}$ is defined for all values of $x \neq 1$, which causes division by zero. There is an infinite variety of functions with restricted domains, but only two types of restricted domains appear on the GRE: division by zero and even roots of negative numbers. For example, the function $f(\ ) = \sqrt{\ -2}$ is defined only if $x - 2 \geq 0$, or $x \geq 2$. The two types of restrictions can be combined. For example, $f(\ ) = \frac{1}{\sqrt{\ -2}}$. Here, $x - 2 \geq 0$ since it's under the square root symbol. Further $x - 2 \neq 0$, or $x \neq 2$, because that would cause division by zero. Hence, the domain is all $x > 2$.

➢ The ***an*** **of a function is the set of values that are assi ned to the *x* values in the do ain.**

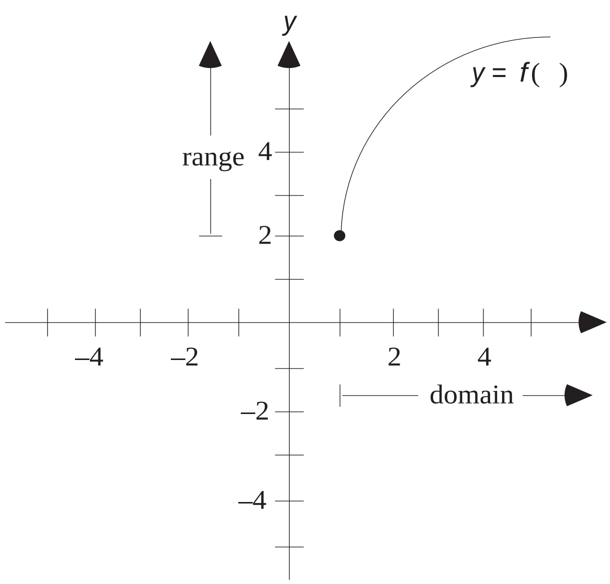
For example, the range of the function $y = f(\ ) = {}^2$ is $y \geq 0$ since a square is never negative. The range of the function $y = f(\ ) = {}^2\ \ 1$ is $y \geq 1$ since $x^2 + 1 \geq 1$. ou can always calculate the range of a function algebraically, but is usually better to graph the function and read off its range from the *y* values of the graph.

GRA

The graph of a function is the set of ordered pairs (x, $f(x)$), where x is in the domain of f and $y = f(x)$.



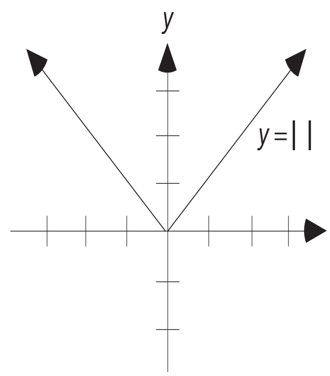
For this function, the domain is all *x* and the range is all $y \geq 0$ (since the graph touches the *x* axis at the origin and is above the *x* axis elsewhere).



For this function, the domain is all $x \geq 1$ and the range is all $y \geq 2$.

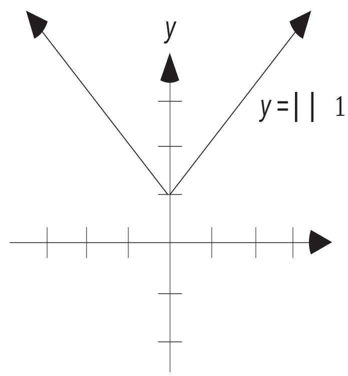
TRAN ATION OF GRA

Many graphs can be obtained by shifting a base graph around by adding positive or negative numbers to various places in the function. Take for example, the absolute value function $y = |\ |$. Its graph is



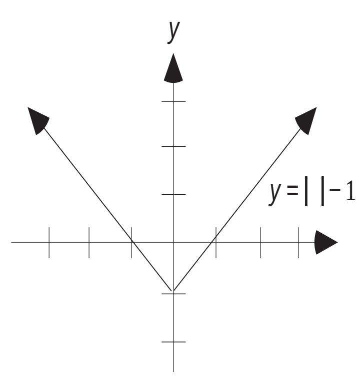
(Notice that sometimes an arrow is added to a graph to indicate the graph continues indefinitely and sometimes nothing is used. To indicate that a graph stops, a dot is added to the terminal point of the graph. Also, notice that the domain of the absolute value function is all because you can take the absolute value of any number. The range is $y \geq 0$ because the graph touches the axis at the origin, is above the axis elsewhere, and increases indefinitely.)

To shift this base graph up one unit, we add 1 outside the absolute value symbol, $y = |\ |\ \ 1$:



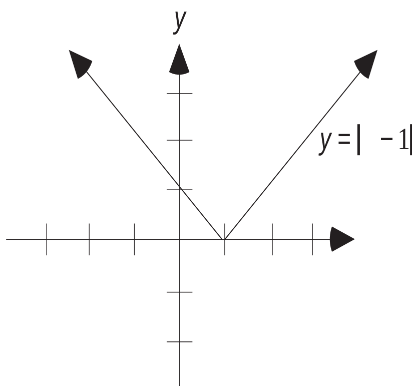
(Notice that the range is now $y \geq 1$.)

To shift the base graph down one unit, we subtract 1 outside the absolute value symbol, $y = |\ | - 1$:



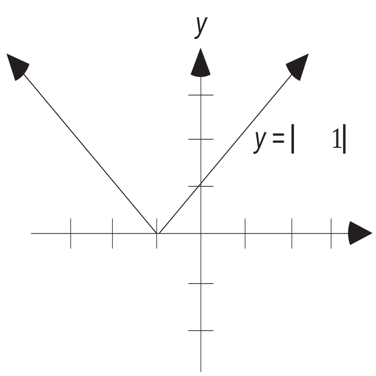
(Notice that the range is now $y \geq -1$.)

To shift the base graph to the right one unit, we subtract 1 inside the absolute value symbol, $y = |\quad - 1|$:



(Notice that the range did not change it's still $y \geq 0$. Notice also that subtracting 1 moved the graph to right. Many students will mistakenly move the graph to the left because that's where the negative numbers are.)

To shift the base graph to the left one unit, we add 1 inside the absolute value symbol, $y = |\quad 1|$:



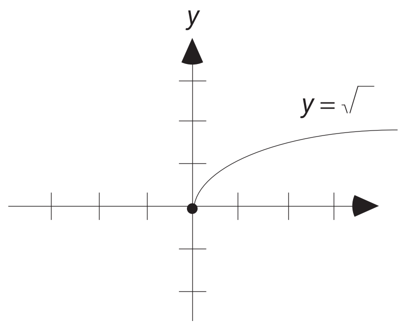
(Notice that the range did not change it's still $y \geq 0$. Notice also that adding 1 moved the graph to left. Many students will mistakenly move the graph to the right because that's where the positive numbers are.)

The pattern of the translations above holds for all functions. o to move a function $y = f(x)$ up c units, add the positive constant c to the exterior of the function: $y = f(x) + c$. To move a function $y = f(x)$ to the right c units, subtract the constant c in interior of the function: $y = f(x - c)$. To summarize, we have

To shift up c units:	$y = f(x) + c$
To shift down c units:	$y = f(x) - c$
To shift to the right c units:	$y = f(x - c)$
To shift to the left c units:	$y = f(x + c)$

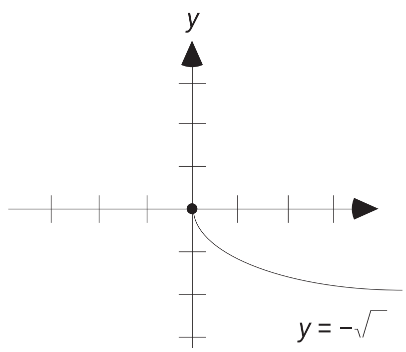
REF E TION OF GRA

Many graphs can be obtained by reflecting a base graph by multiplying various places in the function by negative numbers. Take for example, the square root function $y = \sqrt{\ }$. Its graph is



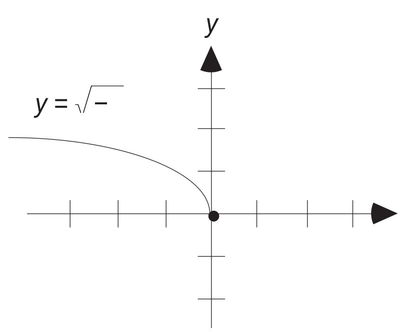
(Notice that the domain of the square root function is all ≥ 0 because you cannot take the square root of a negative number. The range is $y \geq 0$ because the graph touches the axis at the origin, is above the axis elsewhere, and increases indefinitely.)

To reflect this base graph about the axis, multiply the exterior of the square root symbol by negative one, $y = -\sqrt{\ }$:



(Notice that the range is now $y \leq 0$ and the domain has not changed.)

To reflect the base graph about the *y* axis, multiply the interior of the square root symbol by negative one, $y = \sqrt{-\ }$:



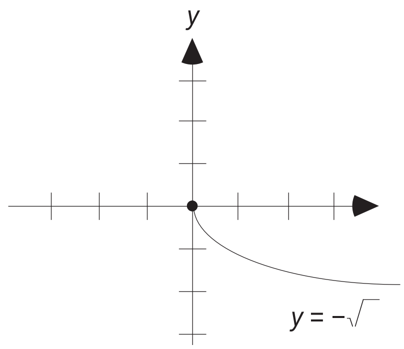
(Notice that the domain is now ≤ 0 and the range has not changed.)

The pattern of the reflections above holds for all functions. o to reflect a function $y = f(\ )$ about the axis, multiply the exterior of the function by negative one: $y = -f(\ )$. To reflect a function $y = f(\ )$ about the *y* axis, multiply the exterior of the function by negative one: $y = f(-\ )$. To summarize, we have

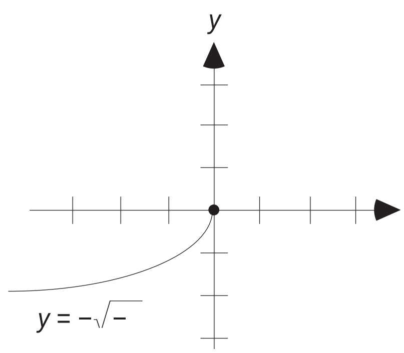
To reflect about the axis: $y = -f(\ )$

To reflect about the *y* axis: $y = f(-\ )$

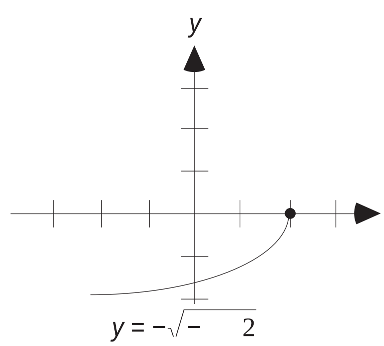
Reflections and translations can be combined.   et's reflect the base graph of the square root function $y = \sqrt{\ }$ about the x axis, the y axis and then shift it to the right 2 units and finally up 1 unit:



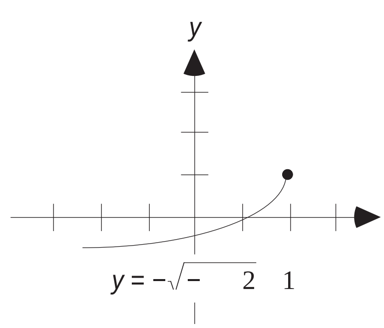
(Notice that the domain is still $x \geq 0$ and the range is now $y \leq 0$.)



(Notice that the domain is now $x \leq 0$ and the range is still y ≤ 0.)



(Notice that the domain is now $x \leq 2$ and the range is still $y \leq 0$.)



(Notice that the domain is still $x \leq 2$ and the range is now $y \leq 1$.)

E A ATION AND OM O ITION OF F N TION

E A ATI N

We have been using the function notation $f(\ )$ intuitively we also need to study what it actually means. ou can think of the letter f in the function notation $f(\ )$ as the name of the function. Instead of using the equation $y = \ ^3 - 1$ to describe the function, we can write $f(\ ) = \ ^3 - 1$. Here, f is the name of the function and $f(\ )$ is the value of the function at . o $f(2) = 2^3 - 1 = 8 - 1 = 7$ is the value of the function at 2. As you can see, this notation affords a convenient way of prompting the evaluation of a function for a particular value of .

Any letter can be used as the independent variable in a function. o the above function could be written $f(\ ) = \ ^3 - 1$. This indicates that the independent variable in a function is ust a placeholder. The function could be written without a variable as follows:

$$f(\ ) = (\ )^3 - 1$$

In this form, the function can be viewed as an input output operation. If 2 is put into the function $f(2)$, then $2^3 - 1$ is returned.

In addition to plugging numbers into functions, we can plug expressions into functions. lugging $y + 1$ into the function $f(\ ) = \ ^2 - \ $ yields

$$f(y \quad 1) = (y \quad 1)^2 - (y \quad 1)$$

ou can also plug other expressions in terms of into a function. lugging 2 into the function $f(\ ) = \ ^2 - \ $ yields

$$f(2\ ) = (2\ )^2 - 2$$

This evaluation can be troubling to students because the variable in the function is being replaced by the same variable. ut the in function is ust a placeholder. If the placeholder were removed from the function, the substitution would appear more natural. In $f(\ ) = (\ )^2 - (\ )$, we plug 2 into the left side $f(2\ )$ and it returns the right side $(2x)^2 - 2x$.

M ITI N

We have plugged numbers into functions and expressions into functions now let's plug in other functions. ince a function is identified with its expression, we have actually already done this. In the example above with $f(\ ) = \ ^2 - \ $ and 2 , let's call 2 by the name ( ). In other words, ( ) = 2 . Then the composition of f with (that is plugging into f) is

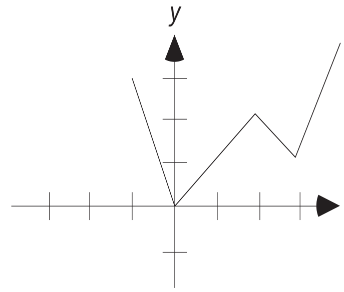
$$f(\ (\ )) = f(2\ ) = (2\ )^2 - 2$$

ou probably won't see the notation $f(\ (x))$ on the test. ut you probably will see one or more problems that ask you perform the substitution. For another example, let $f(\ ) = \frac{1}{\ \ 1}$ and let $(\ ) = \ ^2$. Then $f(\ (\ )) = \frac{1}{\ ^2\ \ 1}$ and $(f(\ )) = \left(\frac{1}{\ +1}\right)^2$. nce you see that the composition of functions merely substitutes one function into another, these problems can become routine. Notice that the composition operation $f(\ (\ ))$ is performed from the inner parentheses out, not from left to right. In the operation $f(\ (2))$, the number 2 is first plugged into the function and then that result is plugged in the function f.

A function can also be composed with itself. That is, substituted into itself. et $f(\ ) = \sqrt{\ } - 2$. Then $f(f(\ )) = \sqrt{\sqrt{\ } - 2} - 2$.

a le The graph of $y = f(x)$ is shown to the right. If $f(-1) = v$, then which one of the following could be the value of $f(v)$

(A) 0
( ) 1
( ) 2
( ) 2.5
(E)

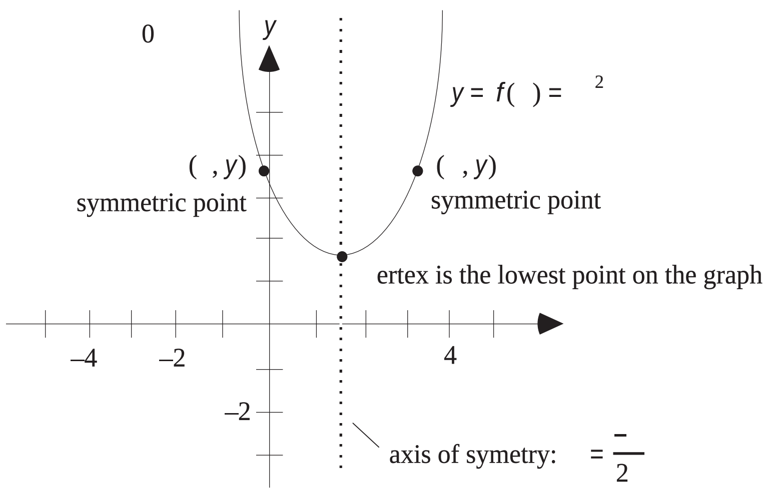


ince we are being asked to evaluate $f(v)$ and we are told that $v = f(-1)$, we are ust being asked to compose $f(x)$ with itself. That is, we need to calculate $f(f(-1))$. From the graph, $f(-1) =$. o $f(f(-1)) = f($ $)$. Again, from the graph, $f($ $) = 1$. o $f(f(-1)) = f($ $) = 1$. The answer is ( ).

ADRATI F N TION

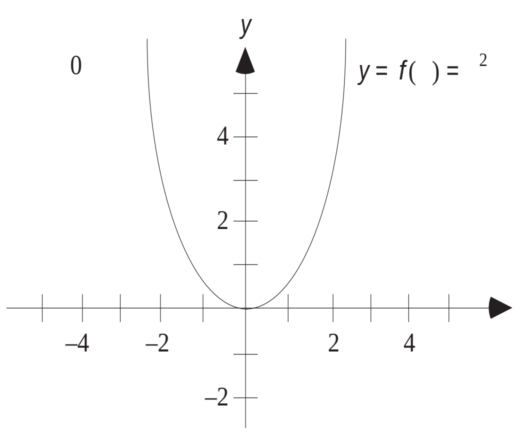
uadratic functions (parabolas) have the following form:

$$y = f(\ ) = \ ^2$$

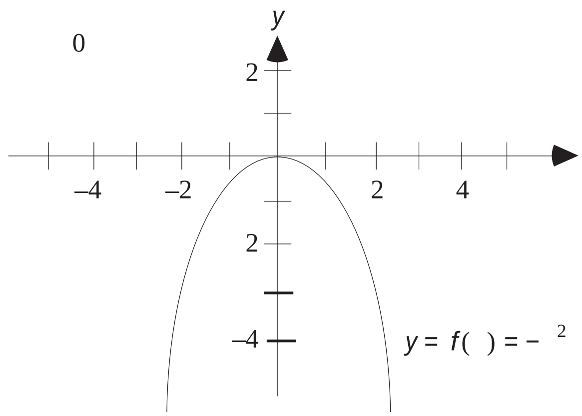


The lowest or highest point on a quadratic graph is called the vertex. The x–coordinate of the vertex occurs at $= \frac{-}{2}$. This vertical line also forms the axis of symmetry of the graph, which means that if the graph were folded along its axis, the left and right sides of the graph would coincide.

In graphs of the form $y = f(\ ) = \ ^2$ if $a > 0$, then the graph opens up.



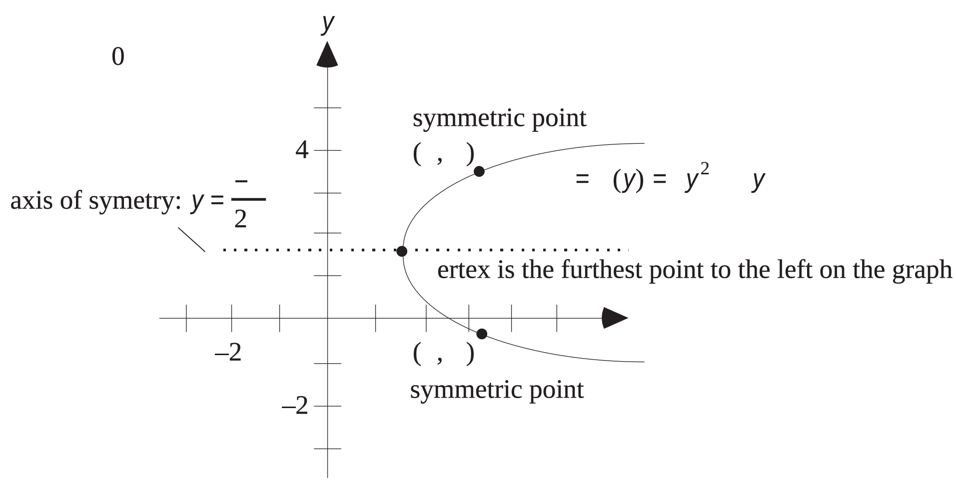
If a 0, then the graph opens down.



y completing the square, the form $y = \quad ^2$ can be written as $y = \quad - \quad ^2$. ou are not expected to know this form on the test. ut it is a convenient form since the vertex occurs at the point (h, k) and the axis of symmetry is the line $x = h$.

We have been analyzing quadratic functions that are vertically symmetric. Though not as common, quadratic functions can also be horizontally symmetric. They have the following form:

$$= \ (y) = \ y^2 \quad y$$



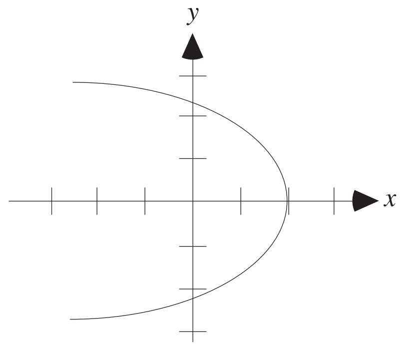
The furthest point to the left on this graph is called the vertex. The y coordinate of the vertex occurs at $y = \frac{-b}{2a}$. This horizontal line also forms the axis of symmetry of the graph, which means that if the graph were folded along its axis, the top and bottom parts of the graph would coincide.

In graphs of the form $= \ y^2 \quad y$ if $a > 0$, then the graph opens to the right and if a 0 then the graph opens to the left.

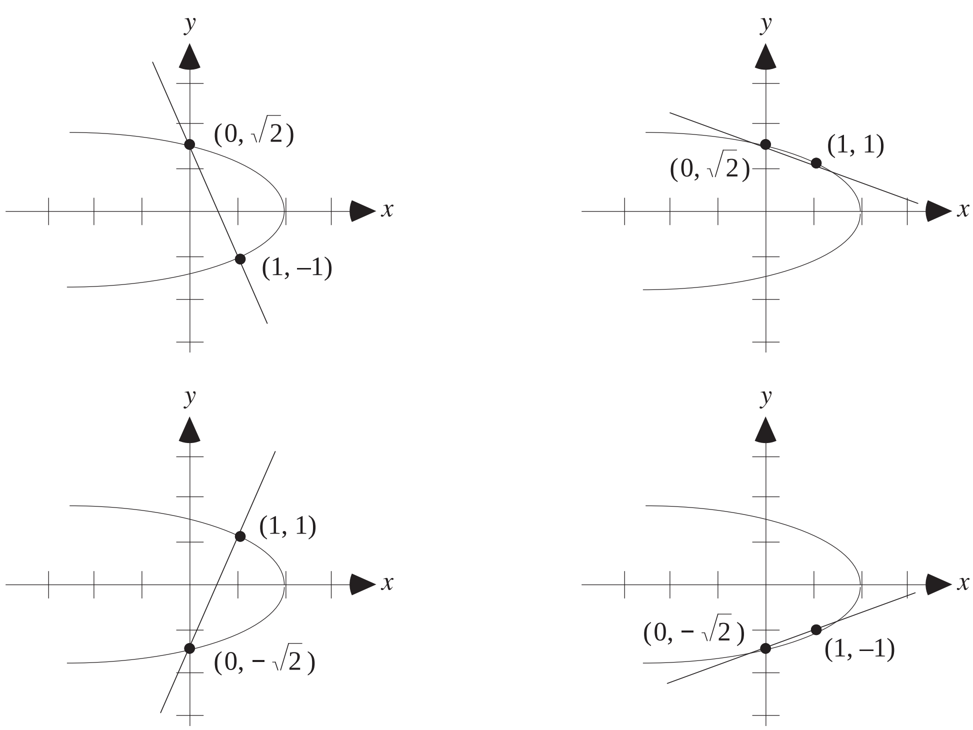
a le The graph of $x = -y^2 \quad 2$ and the graph of the line k intersect at (0, p) and (1, q). Which one of the following is the smallest possible slope of line k

(A) $-\sqrt{2}-1$
( ) $-\sqrt{2} \quad 1$
( ) $\sqrt{2}-1$
( ) $\sqrt{2} \quad 1$
(E) $\sqrt{2} \quad 2$

et's make a rough sketch of the graphs. Expressing $x = -y^2 \quad 2$ in standard form yields $x = -1y^2 \quad 0 \cdot y \quad 2$. ince a = –1, b = 0, and c = 2, the graph opens to the left and its vertex is at (2, 0).



ince p and q can be positive or negative, there are four possible positions for line k (the y coordinates in the graphs below can be calculated by plugging x = 0 and x =1 into the function $x = -y^2 + 2$):



ince the line in the first graph has the steepest negative slope, it is the smallest possible slope. alculating the slope yields

$$m = \frac{\sqrt{2}-(-1)}{0-1} = \frac{\sqrt{2} \quad 1}{-1} = - \ \sqrt{2} \quad 1 \ = -\sqrt{2}-1$$

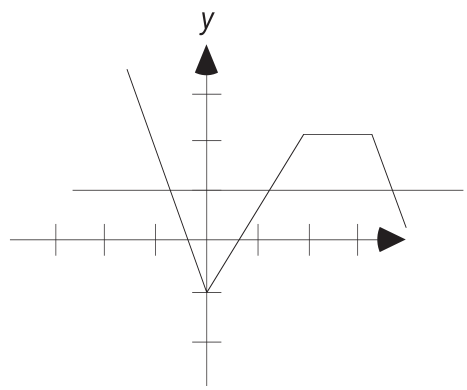
The answer is (A).

A ITATI E E A IOR OF GRA AND F N TION

In this rather vague category, you will be asked how a function and its graph are related. ou may be asked to identify the zeros of a function based on its graph. The zeros, or roots, of a function are the coordinates of where it crosses the axis. r you may be given two graphs and asked for what values are their functions equal. The functions will be equal where they intersect.

a le The graphs of $y = f(\ )$ and $y = 1$ are shown to the right. For how many values does $f(\ )$ equal 1

(A) 0
( ) 1
( ) 2
( )
(E) 4



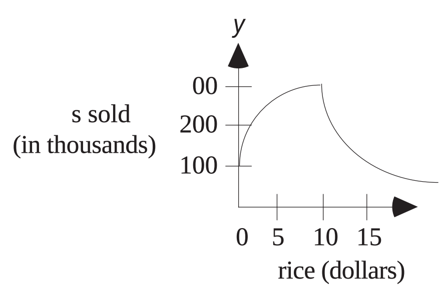
The figure shows that the graphs intersect at three points. At each of these points, both graphs have a height, or y coordinate, of 1. The points are approximately (–. , 1), (1.2, 1), and (4, 1). Hence, $f(\ ) = 1$ for three values. The answer is ( ).

F N TION A MODE OF REA - IFE IT ATION

Functions can be used to predict the outcomes of certain physical events or real life situations. For example, a function can predict the maximum height a pro ectile will reach when fired with an initial velocity, or the number of movie tickets that will be sold at a given price.

a le The graph to the right shows the number of music s sold at various prices. At what price should the s be marked to sell the maximum number of s

(A) 0
( ) 5
( ) 10
( ) 15
(E) 20



As you read the graph from left to right, it shows that sales initially increase rapidly and then slow to a maximum of about 00,000. From there, sales drop precipitously and then slowly approach zero as the price continues to increase. From the graph, sales of 00,000 units on the y axis correspond to a price of about 10 on the axis. The answer is ( ).